

Compactification on the Ω -background and the AGT correspondence

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ABSTRACT: The six-dimensional $(2,0)$ theory formulated in the Ω -background gives rise to two-dimensional effective degrees of freedom. By compactifying the theory on the circle fibers of two cigar-like manifolds, we find that a natural candidate for the effective theory is a chiral gauged WZW model. The symmetry algebra of the model contains the W-algebra that appears on the two-dimensional side of the AGT correspondence. We show that the expectation values of its currents determine the Seiberg-Witten curve of the four-dimensional side.

KEYWORDS: Supersymmetric Gauge Theory, Conformal and W Symmetry, Field Theories in Higher Dimensions

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1 Introduction

The AGT correspondence [1] relates $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions and conformal field theories with W-algebra symmetry in two dimensions [2–8]. Various objects from the two sides are identified through this correspondence, such as the Nekrasov partition functions and conformal blocks [1], the generators of the chiral rings and the currents of the W-algebras [9], and the Seiberg-Witten curves and the expectation values of the W-currents [1]. Perhaps more fundamentally, the correspondence is also manifested in the existence of a W-algebra action on the equivariant cohomology of the instanton moduli space [10, 11].

It has been argued that the connection between these seemingly distant theories originates from six dimensions. The starting point is the $(2, 0)$ theory on $M \times C$, with M a four-manifold and C a punctured Riemann surface, and codimension-two defect operators placed at the punctures of C . For a general choice of the product metric supersymmetry is completely broken, but one can twist the theory to save one of the sixteen supercharges; call it Q . When this is done, the theory is expected to become topological along M and holomorphic along C in the Q -invariant sector. If one compactifies this twisted $(2, 0)$ theory on C , one gets an $\mathcal{N} = 2$ gauge theory on M [12–14] with the familiar Donaldson-Witten twist [15]. If one compactifies the theory instead on M , then one ends up with a twisted $\mathcal{N} = (0, 2)$ supersymmetric theory on C , which in the present case will be a chiral conformal field theory. In the twisted theory, physical quantities are protected under rescaling of the metric of M or C . Then, comparing the effective descriptions of protected quantities leads to a correspondence between the four- and two-dimensional theories.

This argument ignores a crucial point, however: the AGT correspondence does not deal with $\mathcal{N} = 2$ gauge theories of the standard type with M compact, but rather involves the Ω -deformation [16] of them with M noncompact. Thus, one must really consider the situation where the above setup is subject to a deformation that reduces to the Ω -deformation upon compactification on C . In the works [17–19] where the central charges of the effective

conformal field theories were computed from anomalies of the $(2, 0)$ theory, the effect of the Ω -deformation was incorporated by replacing anomaly polynomials by their equivariant counterparts. The success of this procedure indicates that such a deformation does exist. In fact, an M-theory construction has been proposed recently [20].

The goal of this paper is to understand how the expected conformal field theories arise at low energies in the case $M = \mathbb{R}^4$, assuming that there is a formulation of the twisted $(2, 0)$ theory in the Ω -background in the sense just described.

In the standard four-dimensional formulation, the Ω -deformation confines quantum effects near the origin of \mathbb{R}^4 , within a region whose characteristic scale is set by the deformation parameters. So by taking the parameters to be large compared to the energy scale of interest, one can localize quantum effects to the origin. One may call this procedure “compactification on the Ω -background.” If one applies the same procedure to the $(2, 0)$ theory on $\mathbb{R}^4 \times C$, one should obtain an effective theory on C describing degrees of freedom living at the origin of \mathbb{R}^4 .

To identify this effective theory, we perform a different compactification. Exploiting the quasi-topological nature of the twisted theory, we bend \mathbb{R}^4 into the product $D_1 \times D_2$ of two cigar-like manifolds, each consisting of a semi-infinite cylinder capped with a hemisphere on one end, and then take the radii of the cigars to be small. A peculiar feature of this geometry is that on the flat cylinder region the Ω -deformation can be canceled by a change of variables [21]. This property allows us to represent the effect of the Ω -deformation by the insertion of Q -invariant operators, supported on codimension-two submanifolds located at the tips of the cigars. We describe how this works in section 2.

In the absence of the Ω -deformation, compactification on the circle fibers of the cigars gives $\mathcal{N} = 4$ super Yang-Mills theory on $L_1 \times L_2 \times C$, where L_1 and L_2 are respectively the axes of D_1 and D_2 , each a half-line $[0, \infty)$. This four-manifold with a corner has two boundary components intersecting orthogonally, $\{0\} \times L_2 \times C$ coming from the tip of D_1 and $L_1 \times \{0\} \times C$ coming from the tip of D_2 . These boundaries are endowed with half-BPS boundary conditions related by S -duality. We determine these boundary conditions in section 3.

Turning on the Ω -deformation translates in the compactified theory to introducing Q -invariant boundary terms to the action. This may sound strange. After all, the Ω -deformation is expected to give rise to two-dimensional dynamics, not three. Shouldn’t it then produce something defined on a two-dimensional submanifold? Quite the contrary, these boundary couplings generate exactly such dynamics, because the two boundaries themselves have a common boundary which is two-dimensional. The boundary couplings must satisfy certain criteria derived from the quasi-topological invariance in six dimensions. In section 4, we will find that natural candidates are a Chern-Simons term for the complexified gauge group and its S -dual, which, formulated on a manifold with boundary, induce boundary degrees of freedom described by a gauged WZW model [22–24]. The emergence of dynamical boundary degrees of freedom from otherwise topological Chern-Simons theory is a prototypical example of holography.

These degrees of freedom come from six dimensions since compactification cannot create new ones. In turn, they should be the degrees of freedom seen at low energies of the $(2, 0)$ theory formulated in the Ω -background — the intersection of the two boundaries is

nothing but the origin of \mathbb{R}^4 ! Therefore, we conclude that the $(2,0)$ theory “compactified on the Ω -background” is described by the gauged WZW model. The purely bosonic nature of the latter theory solves an apparent puzzle that conformal field theories relevant for the AGT correspondence for $M = \mathbb{R}^4$ are not supersymmetric as the naive compactification argument may suggest.

Besides, the model has the right property: its symmetry algebra contains a W-algebra, precisely the one that appears in the AGT correspondence. Furthermore, one can relate the W-algebra quite naturally to the Seiberg-Witten curve of the effective $\mathcal{N} = 2$ theory, and the relation agrees with the conjectured one. These facts, explained more fully in section 5, provide evidence for the correctness of our argument.

Our conclusion reinforces the idea that the Q -cohomology of the $(2,0)$ theory contains a W-algebra [25], albeit Q here should probably be replaced by a different supercharge \tilde{Q} appropriate for the Ω -deformed situation. Up in six dimensions, the \tilde{Q} -cohomology of states is naturally a module over the \tilde{Q} -cohomology of operators. Once we go down to four dimensions, the action of some subalgebra may still be present, but then the way it is realized is probably not very obvious. The AGT correspondence seems to be an example of such an instance: the \tilde{Q} -cohomology in six dimensions contains the W-algebra, and it acts on the \tilde{Q} -cohomology of the Ω -deformed $\mathcal{N} = 2$ theory — or the equivariant cohomology of the instanton moduli space — in a nontrivial way.

2 Localizing the Ω -deformation

To begin, let us explain how the $(2,0)$ theory on $M \times C$ is twisted, and why the twisted theory is expected to be topological along M and holomorphic along C .

The general idea of twisting is to replace the holonomy group by a combination of the holonomy and R-symmetry groups under which a fraction of the supercharges transform as scalars, therefore are left unbroken by the curvature. For the $(2,0)$ theory on $M \times C$, the holonomy is reduced to

$$\text{Spin}(4) \times \text{Spin}(2) \cong \text{SU}(2)_\ell \times \text{SU}(2)_r \times \text{U}(1)_C, \quad (2.1)$$

and the R-symmetry is $\text{Spin}(5)$. We split the latter as

$$\text{Spin}(3) \times \text{Spin}(2) \cong \text{SU}(2)_R \times \text{U}(1)_{\mathcal{R}}. \quad (2.2)$$

The supercharges transform as $\mathbf{4}_+ \otimes \mathbf{4}$ under $\text{Spin}(6) \times \text{Spin}(5)$, where $\mathbf{4}_+$ is a positive chirality spinor of $\text{Spin}(6)$ and $\mathbf{4}$ is a spinor of $\text{Spin}(5)$. Under the above subgroups of $\text{Spin}(6)$ and $\text{Spin}(5)$, they decompose as

$$\mathbf{4}_+ \otimes \mathbf{4} \rightarrow ((\mathbf{2}, \mathbf{1})_{1/2} \oplus (\mathbf{1}, \mathbf{2})_{-1/2}) \otimes (\mathbf{2}_{1/2} \oplus \mathbf{2}_{-1/2}). \quad (2.3)$$

We replace $\text{SU}(2)_r$ by the diagonal subgroup $\text{SU}(2)'_r$ of $\text{SU}(2)_r \times \text{SU}(2)_R$, and $\text{U}(1)_C$ by the diagonal subgroup $\text{U}(1)'_C$ of $\text{U}(1)_C \times \text{U}(1)_{\mathcal{R}}$. Under $\text{SU}(2)_\ell \times \text{SU}(2)'_r \times \text{U}(1)'_C$, the supercharges transform as

$$(\mathbf{2}, \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{3})_{-1}. \quad (2.4)$$

We see that the twisting produces one scalar supercharge, which we call Q . From the viewpoint of M , this is the Donaldson-Witten twist [15] of $\mathcal{N} = 2$ supersymmetry. From the viewpoint of C , it is the unique twist of $\mathcal{N} = (0, 2)$ supersymmetry.

In the case of twisted $\mathcal{N} = 2$ gauge theories in four dimensions, a similar supercharge obeys $Q^2 = 0$ up to a gauge transformation, and upon restricting to Q - and gauge-invariant operators and states, physical quantities depend only on the Q -cohomology classes of operators and states involved. Meanwhile, the energy-momentum tensor is Q -exact. It follows that the twisted theories are topological, in the sense that physical quantities are invariant under deformations of the metric. Likewise, for twisted $\mathcal{N} = (0, 2)$ theories in two dimensions, the components of the energy-momentum tensor generating antiholomorphic reparametrizations vanish in Q -cohomology. Thus, the twisted theories are holomorphic. The $(2, 0)$ theory reduces to theories of these kinds by appropriate compactification on C or M , so we expect that the twisted $(2, 0)$ theory becomes topological along M and holomorphic along C if we think of Q as a BRST operator.

This expectation is backed up by the existence of an analogous twist in four dimensions, studied by Kapustin in [26]. Kapustin's twist can be applied to any $\mathcal{N} = 2$ gauge theory with nonanomalous $SU(2) \times U(1)$ R-symmetry group, formulated on the product $\Sigma \times C$ of Riemann surfaces. Such theories are superconformal. We write the $U(1)$ factor of $SU(2) \times U(1)$ as $U(1)_R$, to distinguish from a maximal torus of the $SU(2)$ subgroup which we denote by $U(1)_R$. The holonomy group of $\Sigma \times C$ is $U(1)_\Sigma \times U(1)_C$. One replaces $U(1)_\Sigma$ with the diagonal subgroup of $U(1)_\Sigma \times U(1)_R$. Similarly, one twists $U(1)_C$ with $U(1)_R$. After twisting, two of the eight supercharges, Q_ℓ and Q_r , are scalars. For the BRST operator one takes a linear combination

$$Q = Q_\ell + tQ_r \quad (2.5)$$

with $t \in \mathbb{C}^\times$. Then, the twisted theory is topological along Σ and holomorphic along C . For $t = 0$ or ∞ , it is holomorphic both on Σ and C .

The relevance of Kapustin's twist to the present story is that if one takes $M = \Sigma \times \Sigma'$, with Σ' another Riemann surface, and compactifies the twisted $(2, 0)$ theory on Σ' , one can obtain an $\mathcal{N} = 2$ superconformal theory on $\Sigma \times C$. The complex structure of Σ' determines the complexified gauge couplings which combine the gauge couplings and the θ -angles. If the twisted theory has the claimed quasi-topological property, then $\mathcal{N} = 2$ theories so obtained have to be twisted in such a way that they are topological along Σ and holomorphic along C , and moreover independent of the couplings. Kapustin's twist does produce theories with the desired properties, and it is essentially the only such twist. The $U(1)_R$ symmetry of the $(2, 0)$ theory is the $U(1)_R$ group used in Kapustin's twist, while a maximal torus of $SU(2)_R$ is the $U(1)_R$ group.

Having defined the twisted $(2, 0)$ theory on $M \times C$, we now want to turn on the Ω -deformation. Let us first recall the standard formulation in four dimensions.

The Ω -deformation of an $\mathcal{N} = 2$ gauge theory on a four-manifold M can be considered when M admits a $U(1)$ isometry. Denote by V the vector field generating the isometry. Then, the procedure is roughly to replace the adjoint scalar ϕ in the vector multiplet as

$$\phi \rightarrow \phi + \epsilon V^\mu D_\mu, \quad (2.6)$$

where ϵ is a complex parameter and $D = d + A$ is the covariant derivative. Since ϕ is replaced by a differential operator, this is not a change of variables but a deformation of the theory.

Like the undeformed case, the Ω -deformed theory is topological after twisting, but with respect to a different supercharge. The twisted supercharges are a scalar Q , a one-form Q_μ , and a self-dual two-form $Q_{\mu\nu}$. Of these, at least Q and $V^\mu Q_\mu$ are unbroken in the undeformed theory. The Ω -deformation preserves the linear combination

$$\tilde{Q} = Q + \epsilon V^\mu Q_\mu. \quad (2.7)$$

To make the Ω -deformed theory topological one takes \tilde{Q} as the BRST operator.

We will consider the case $M = \mathbb{R}^4$. In this case we can deform the theory with two commuting $U(1)$ isometries, rotating two orthogonal two-planes in \mathbb{R}^4 . To do this, we just replace ϵV in the above formulas by $\epsilon_1 V_1 + \epsilon_2 V_2$, where V_1 and V_2 are the Killing vector fields for the rotations, and ϵ_1 and ϵ_2 are the corresponding parameters.

As stressed already, we will assume that the Ω -deformation of a twisted $\mathcal{N} = 2$ gauge theory lifts to a deformation of the underlying twisted $(2, 0)$ theory. Later we will rephrase our assumption in somewhat different terms.

To better understand the effect of the Ω -deformation to the twisted $(2, 0)$ theory, we want to compactify the theory to lower dimensions where a Lagrangian description is available. To this end we equip \mathbb{R}^4 with the metric for $D_1 \times D_2$, the product of two cigars whose radii ρ_1 and ρ_2 we will take to be small. It turns out that the Ω -deformation has a particularly nice description in this setup, thanks to the following property of the cigar geometry observed by Nekrasov and Witten [21]: the Ω -deformation on a cigar can be canceled by a change of variables everywhere except in the curved region near the tip. Here we briefly explain why this is true.

Consider an $\mathcal{N} = 2$ gauge theory on the product of a two-manifold and a flat cylinder of radius ρ . We turn on the Ω -deformation using the rotation about the axis of the cylinder. Since the $U(1)_R$ symmetry rotates ϵ by a phase, we can assume that ϵ is real. Then, the Ω -deformation is realized by the substitution

$$A_4 \rightarrow A_4 + \epsilon \rho D_1, \quad (2.8)$$

where A_4 is defined by writing $\phi = A_4 + iA_5$ using antihermitian adjoint-valued one-forms A_4 and A_5 , and $x^1 \sim x^1 + 2\pi\rho$ is the coordinate in the circle direction of the cylinder.

Let us combine this with the rescaling of the radius

$$\rho \rightarrow \rho \sqrt{1 + \epsilon^2 \rho^2}, \quad (2.9)$$

and introduce the rescaled coordinate $\hat{x}^1 = x^1 / \sqrt{1 + \epsilon^2 \rho^2}$ of the same periodicity as before. Also, make the change of variables $\hat{A}_4 = A_4 / \sqrt{1 + \epsilon^2 \rho^2}$ and $\hat{A}_1 = A_1 + \epsilon \rho \hat{A}_4$, and define $\hat{D}_1 = \partial/\partial \hat{x}^1 + \hat{A}_1$. In terms of these, the effect of the combined operations (2.8) and (2.9) is to replace

$$A_4 \rightarrow \frac{1}{\sqrt{1 + \epsilon^2 \rho^2}} (\hat{A}_4 + \epsilon \rho \hat{D}_1), \quad (2.10)$$

while we have

$$D_1 = \frac{1}{\sqrt{1 + \epsilon^2 \rho^2}} (\hat{D}_1 - \epsilon \rho \hat{A}_4). \quad (2.11)$$

The point is that the two substitutions (2.10) and (2.11) together can be thought of as a rotation in the A_4 - D_1 plane, and under such a rotation (followed by the corresponding rotation of fermions) the action of the undeformed theory is invariant. In other words, the Ω -deformation on a cigar can be undone by a change of variables and a rescaling of the radius. The undeformed theory so obtained has the coupling constant rescaled as

$$e^2 \rightarrow \frac{e^2}{\sqrt{1 + \epsilon^2 \rho^2}}, \quad (2.12)$$

since the volume form contains the factor $dx^1 = \sqrt{1 + \epsilon^2 \rho^2} d\hat{x}^1$.

One can replace the cylinder by a cigar, and apply the same argument in the region where the cigar looks like a flat cylinder. Then one sees that the Ω -deformation can be canceled away from the tip of the cigar, by a change of variables. This was the insight of Nekrasov and Witten.

We can readily extend the above argument to the situation in which the theory is formulated on $D_1 \times D_2$. Taking the coordinates in the circle directions to be $x^1 \sim x^1 + 2\pi\rho_1$ and $x^2 \sim x^2 + 2\pi\rho_2$, away from the tips of the cigar the Ω -deformation is given by the substitution

$$A_4 \rightarrow A_4 + \epsilon_1 \rho_1 D_1 + \epsilon_2 \rho_2 D_2. \quad (2.13)$$

This is the same operation as the Ω -deformation (2.8) for a single cylinder if we set $(\epsilon\rho)^2 = (\epsilon_1\rho_1)^2 + (\epsilon_2\rho_2)^2$ and rotate the x^1 - x^2 plane by angle $\tan^{-1}(\epsilon_1\rho_1/\epsilon_2\rho_2)$. The rotation does not change the form of the action or the standard flat metric, so the Ω -deformation can be canceled¹ by a change of variables together with a rescaling of the metric and the coupling by a factor of $1/\sqrt{1 + \epsilon^2 \rho^2}$.

Therefore, the Ω -deformed action \tilde{S} , written in nonstandard variables, is equal to the undeformed action S with the rescaled metric and coupling, plus terms that vanish on the flat cylinder region. These terms are \tilde{Q} -invariant, as both \tilde{S} and S are. To put it differently, the effect of the Ω -deformation on $D_1 \times D_2$ is essentially to insert \tilde{Q} -invariant operators supported in the neighborhood of $\{0\} \times D_2$ and $D_1 \times \{0\}$, where $0 \in D_1$ or D_2 is the tip of either cigar.

In the limit where the radii of the cigars are sent to zero, \tilde{Q} reduces to Q and the curvature localizes to the tips. Thus the inserted operators are well approximated by Q -invariant operators supported at the tips, and the approximation becomes arbitrarily good

¹The argument is presented here for the case when ϵ_1 and ϵ_2 are both real. The general case can be treated as follows. The general Ω -deformation is given by the substitutions $A_4 \rightarrow A_4 + \text{Re } \epsilon_1 \rho_1 D_1 + \text{Re } \epsilon_2 \rho_2 D_2$ and $A_5 \rightarrow A_5 + \text{Im } \epsilon_1 \rho_1 D_1 + \text{Im } \epsilon_2 \rho_2 D_2$. If $\theta_1 = \arg \epsilon_1$ and $\theta_2 = \arg \epsilon_2$ are equal, one can make ϵ_1 and ϵ_2 both real by a $U(1)_R$ rotation. So suppose $\theta_1 \neq \theta_2$. The Ω -deformation can be canceled if these substitutions can be put into the form $A_4 \rightarrow A_4 + \epsilon'_1 \rho_1 D_1$ and $A_5 \rightarrow A_5 + \epsilon'_2 \rho_2 D_2$ for some $\epsilon'_1, \epsilon'_2 \in \mathbb{R}$, by rotating the x^1 - x^2 plane. This requires $\text{Re } \epsilon_1 \rho_1 / \text{Re } \epsilon_2 \rho_2 = -\text{Im } \epsilon_2 \rho_2 / \text{Im } \epsilon_1 \rho_1$, or $\sin(2\theta_1) / \sin(2\theta_2) = -|\epsilon_2|^2 \rho_2^2 / |\epsilon_1|^2 \rho_1^2$. Since the function $f(\alpha) = \sin(2\theta_1 + \alpha) / \sin(2\theta_2 + \alpha)$ ranges from $-\infty$ to $+\infty$ as α varies, it is possible to satisfy this condition by a $U(1)_R$ rotation.

as the radii go to zero. Reversing the logic, for twisted $\mathcal{N} = 2$ gauge theories on \mathbb{R}^4 , we may define the Ω -deformation by the insertion of these Q -invariant operators. For this definition coincides with the standard one after \mathbb{R}^4 is deformed to $D_1 \times D_2$, the metric and the coupling are rescaled, and the zero radii limit is taken, the operations none of which affects the topological field theory. (The Kaluza-Klein modes discarded in taking the zero radii limit are not important, as the momenta are identically zero in Q -cohomology.)

Then, the assumption we really want to make is that these Q -invariant operators lift to Q -invariant operators in the twisted $(2, 0)$ theory on $D_1 \times D_2 \times C$, supported at the tips of the cigars, $\{0\} \times D_2 \times C$ and $D_1 \times \{0\} \times C$. The insertion of the lifted operators may be regarded as a definition of the Ω -deformation of the twisted $(2, 0)$ theory on $\mathbb{R}^4 \times C$.

3 Down to four dimensions

The motivation for taking the small radii limit was to compactify the $(2, 0)$ theory to lower dimensions where we understand things better. So let us look at the effective description of the theory.

Compactification on the circle fibers of the cigars gives a theory on the four-manifold with a corner $L_1 \times L_2 \times C$, where L_1 and L_2 are half-lines. The tips of the cigars correspond to the boundaries $\{0\} \times L_2 \times C$ and $L_1 \times \{0\} \times C$. The $(2, 0)$ theory compactified on a torus is $\mathcal{N} = 4$ super Yang-Mills theory whose complexified gauge coupling $\tau = \theta/2\pi + 4\pi i/e^2$ is given by the complex structure modulus of the torus. In our case, away from the tips we are compactifying the theory on a rectangular torus with sides of length $2\pi\rho_1$ and $2\pi\rho_2$. Thus, the bulk theory is $\mathcal{N} = 4$ super Yang-Mills theory with $\tau = i\rho_1/\rho_2$. The gauge group is the compact Lie group G associated with the simply-laced Lie algebra \mathfrak{g} specifying the type of the $(2, 0)$ theory.

The twisting of the $(2, 0)$ theory reduces to the Kapustin twist, which makes the theory topological along $L_1 \times L_2$ and holomorphic along C . The compactified theory has a fixed value of t as the twisted $(2, 0)$ theory carries only one supercharge. The precise value is inessential, however; the $U(1)_R$ symmetry rotates Q_ℓ and Q_r by opposite phases, so one can set t to any value except 0 or ∞ by the complexified $U(1)_R$ -action applied in six dimensions, without changing the Q -cohomology. It is convenient to set $t = i$.

Turning on the Ω -deformation in the twisted $(2, 0)$ theory means inserting Q -invariant operators placed at the tips of the cigars. These operators descend to Q -invariant boundary couplings in the $\mathcal{N} = 4$ theory. Eventually we will deduce what the boundary couplings are, but before doing that, we want to identify the boundary conditions in the absence of the Ω -deformation. For this purpose it is helpful to resort to brane constructions.

We will assume that the cotangent bundle T^*C of C admits a complete Calabi-Yau metric. This is the case if C is a sphere or a torus, for example. For $C = S^2$, one can endow T^*C with the Eguchi-Hanson metric.

First we focus on the boundary created at the tip of D_1 , so let us forget about the cap of D_2 , replacing it with a cylinder \tilde{D}_2 . We denote the axis of \tilde{D}_2 by $\tilde{L}_2 = \mathbb{R}$. We consider N M5-branes wrapped on $D_1 \times \tilde{D}_2 \times \{0\} \times C$ in M-theory on $T^*D_1 \times \tilde{D}_2 \times \mathbb{R} \times T^*C$. If T^*D_1 and T^*C are endowed with Calabi-Yau metrics, so that some of the supersymmetries are

preserved, then the low-energy dynamics of the M5-branes realizes the $(2, 0)$ theory of type A_{N-1} on $D_1 \times \tilde{D}_2 \times C$. With D_1 being a cigar, we can choose T^*D_1 to be a Taub-NUT space TN. This space is a circle fibration over \mathbb{R}^3 such that the radius of the circle shrinks to zero at the origin and approaches a finite asymptotic value at infinity, so the fibers over each radial direction make up a cigar. To embed D_1 in TN, we pick a direction L_1 and identify D_1 with the fibers over L_1 . For clarity we write $\tilde{L}_1 = \mathbb{R}$ and $\mathbb{R}^3 = \tilde{L}_1 \times \mathbb{R}^2$, with $L_1 \subset \tilde{L}_1$.

We compactify this system on the S^1 of $\tilde{D}_2 = \tilde{L}_2 \times S^1$. This gives N D4-branes wrapped on $D_1 \times \tilde{L}_2 \times \{0\} \times C$ in Type IIA string theory on $TN \times \tilde{L}_2 \times \mathbb{R} \times T^*C$. We still want to compactify the system on the circle of D_1 . So we perform a T -duality to unwrap the branes from the circle, and take the radius of the dual circle \hat{S}^1 to be large. At first sight, the T -duality may appear to replace TN by the dual fibration with fiber \hat{S}^1 , and turns the D4-branes to D3-branes. This is actually not the case. Rather, it produces a D3-NS5 system in Type IIB string theory on

$$\hat{S}^1 \times \tilde{L}_1 \times \mathbb{R}^2 \times \tilde{L}_2 \times \mathbb{R} \times T^*C, \quad (3.1)$$

with N D3-branes ending on a single NS5-brane [27, 28]. The support of the D3-branes is

$$\{p\} \times L_1 \times \{0\} \times \tilde{L}_2 \times \{0\} \times C, \quad (3.2)$$

while the NS5-brane is wrapped on

$$\{p\} \times \{0\} \times \{0\} \times \tilde{L}_2 \times \mathbb{R} \times T^*C, \quad (3.3)$$

where p is a point of \hat{S}^1 . The low-energy dynamics of the D3-branes is described by $\mathcal{N} = 4$ super Yang-Mills theory on $L_1 \times \tilde{L}_2 \times C$. We have a half-BPS boundary condition on the boundary located at $0 \in L_1$, where the D3-branes end on the NS5-brane.

Only the local geometry matters to the boundary condition. Thus, if we ignore the curvature of C for simplicity, the above D3-NS5 system leads to the same boundary condition as that for a system of N flat D3-branes ending on a single flat NS5-brane, with the D3- and NS5-branes sharing three spacetime directions. The $\mathcal{N} = 4$ theory on the D3-branes has six scalars coming from the fluctuations in the six normal directions. The rotations in the normal plane give rise to the R-symmetry group $SO(6)$, but the presence of the NS5-brane breaks it to $SO(3)_X \times SO(3)_Y$, the product of the rotation groups of the three-planes tangent and normal to the NS5-brane. The breaking of $SO(6)$ divides the scalars into two triplets, \vec{X} and \vec{Y} , transforming as a vector under $SO(3)_X$ and $SO(3)_Y$, respectively. The NS5-brane imposes Neumann boundary conditions on \vec{X} and the gauge field A ; writing x for the coordinate of L_1 , on the boundary we have

$$D_x \vec{X} = F_{x\mu} = 0, \quad (3.4)$$

where F is the curvature of A . Fluctuations normal to the NS5-brane must vanish on the boundary, so \vec{Y} obey Dirichlet boundary conditions:

$$\vec{Y} = 0. \quad (3.5)$$

For the fermions, the boundary conditions set half of them to zero. Lastly, requiring that the boundary conditions themselves be invariant under the would-be unbroken supersymmetries imposes further conditions on the fermions.

Next, we look at the boundary coming from the tip of D_2 , this time replacing D_1 with a cylinder \tilde{D}_1 with axis $\tilde{L}_1 = \mathbb{R}$. So we start with a system of N M5-branes wrapped on $\tilde{D}_1 \times D_2 \times \{0\} \times C \subset \tilde{D}_1 \times \text{TN}' \times \mathbb{R} \times T^*C$, with TN' another Taub-NUT space, and first compactify it on D_2 . We embed D_2 in TN' as the fibers over a radial direction L_2 in the base $\mathbb{R}^3 = \mathbb{R}^2 \times \tilde{L}_2$. Then, upon compactification, the M5-branes become D4-branes wrapped on $\tilde{D}_1 \times \{0\} \times L_2 \times \{0\} \times C$ in the spacetime $\tilde{D}_1 \times \mathbb{R}^2 \times \tilde{L}_2 \times \mathbb{R} \times T^*C$. In addition, the compactification creates a D6-brane supported at the origin of the base [27, 28], that is, on $\tilde{D}_1 \times \{0\} \times \{0\} \times \mathbb{R} \times T^*C$. After that, we perform a T -duality on the S^1 of $\tilde{D}_1 = S^1 \times \tilde{L}_1$. The end result is a D3-D5 system in the spacetime (3.1), with N D3-branes ending on a D5-brane. The D3-branes are supported on

$$\{p\} \times \tilde{L}_1 \times \{0\} \times L_2 \times \{0\} \times C \quad (3.6)$$

and the D5-brane is on

$$\{p\} \times \tilde{L}_1 \times \{0\} \times \{0\} \times \mathbb{R} \times T^*C. \quad (3.7)$$

The D3-D5 system we have arrived at is the S -dual of the D3-NS5 system. This is consistent with the fact that the S -duality of $\mathcal{N} = 4$ super Yang-Mills theory is realized in six dimensions as the modular transformations acting on the torus on which the $(2, 0)$ theory is compactified. When the torus is rectangular, interchanging the two sides gives the duality $\tau \rightarrow -1/\tau$. In the D3-D5 construction the roles of D_1 and D_2 are switched compared to the D3-NS5 construction, so in effect we have applied S -duality.

At low energies the dynamics of the D3-branes is described by $\mathcal{N} = 4$ super Yang-Mills theory on $\tilde{L}_1 \times L_2 \times C$, with the S -dual half-BPS boundary condition. The presence of the D5-brane breaks the R-symmetry group $\text{SO}(6)$ to $\text{SO}(3)_{X'} \times \text{SO}(3)_{Y'}$, and divides the scalars into two triplets \vec{X}' and \vec{Y}' . Writing y for the coordinate of L_2 , the D3-D5 boundary conditions are such that near the boundary \vec{X}' are approximated by a solution of the Nahm equations

$$\frac{DX'_i}{Dy} + \epsilon_{ijk}[X'_i, X'_j] = 0, \quad (3.8)$$

with the particular singular behavior

$$\vec{X}' = \frac{\vec{t}}{y} + \dots. \quad (3.9)$$

Here \vec{t} is a triplet of elements in \mathfrak{g} giving a principal $\mathfrak{su}(2)$ embedding with the standard commutation relations $[t_i, t_j] = \epsilon_{ijk}t_k$, and the ellipsis refers to terms less singular than $1/y$ [29]. We can gauge away A_y at $y = 0$. The other components of the gauge field obey Dirichlet boundary conditions,² as do the scalars \vec{Y}' :

$$A = \vec{Y}' = 0. \quad (3.10)$$

²If C is curved, the boundary value of the gauge field must actually be related to the Riemannian connection on the boundary in a specific manner [30]. This modification does not affect our analysis.

The boundary condition sets to zero half of the fermions, possibly different from the previous half.

Combining the two descriptions of the compactification, we conclude that the $(2,0)$ theory of type A_{N-1} compactified on the circle fibers of $D_1 \times D_2$ is realized by a D3-D5-NS5 system in Type IIB string theory in the spacetime (3.1), with N D3-branes ending on a D5-brane in one direction and on an NS5-brane in another. The D3-branes are supported on

$$\{p\} \times L_1 \times \{0\} \times L_2 \times \{0\} \times C, \quad (3.11)$$

and the NS5- and D5-branes are wrapped on the same submanifolds (3.3) and (3.7) as before. Notice that the directions normal to the D3- and NS5-branes are the same as those normal to the D3- and D5-branes. Thus $\vec{X} = \vec{X}'$ and $\vec{Y} = \vec{Y}'$. Otherwise, the D3-NS5 and D3-D5 boundary conditions would not be compatible.

The boundary conditions derived from the D3-D5-NS5 system should preserve the supersymmetry of the twisted theory. One may feel that this point is somewhat obscured in the above construction, since we looked at the two boundaries separately. Nonetheless, this must be true, as the following argument shows. Ideally, one could start with a system of M5-branes wrapped on $D_1 \times D_2 \times C \subset V \times T^*C$, where V is a seven-manifold of G_2 holonomy in which $D_1 \times D_2$ is embedded as a supersymmetric cycle. This system has only one unbroken supersymmetry in general, and it is precisely the one that is preserved by the particular twist we applied to the $(2,0)$ theory [31]. Instead of picking V , we started with $TN \times \tilde{D}_2 \times \mathbb{R}$ or $\tilde{D}_1 \times TN' \times \mathbb{R}$ pretending that D_2 or D_1 was a cylinder. These are certainly G_2 manifolds, their holonomy being $SU(2) \subset G_2$. Therefore, the D3-NS5 and D3-D5 systems both preserve the supersymmetry of the twisted theory, hence so does the total D3-D5-NS5 system.

There is a question about which scalar on the D3-branes corresponds to which in the twisted theory. We can determine the correspondence from the fact that the D3-NS5 boundary condition admits a generalization to the case with nonvanishing θ -angle [29]. For $\theta \neq 0$, the simple Neumann boundary conditions (3.4) are modified due to the boundary coupling by a Chern-Simons term, constructed from the complex gauge field given by a linear combination of A and \vec{X} [30]. For such a boundary coupling to make sense when C is curved, \vec{X} must be twisted to a one-form of the type $X_y dy + X_z dz + X_{\bar{z}} d\bar{z}$, where z is a holomorphic coordinate on C . This allows us to identify, up to $SO(3)_X$ rotations and rescalings,

$$\begin{aligned} X_z &= X_1 + iX_2, \\ X_{\bar{z}} &= X_1 - iX_2, \\ X_y &= X_3. \end{aligned} \quad (3.12)$$

As for \vec{Y} , after the twisting one of them becomes a one-form $Y_x dx$ and the others remain to be scalars.

4 Turning on the Ω -deformation

Now we turn on the Ω -deformation, and try to determine the induced Q -invariant boundary couplings and the dynamical degrees of freedom that emerge on “the boundary of the

boundaries.” Since the boundary couplings for the two boundaries should be related by S -duality, we will only consider the one for the D3-NS5 boundary. For simplicity, we will assume that C has no boundary, and write $L \times C$ for $\{0\} \times L_2 \times C$ with L a half-line.

The boundary coupling is constructed out of the gauge field A and the adjoint-valued one-form X , and half of the fermions.³ It must satisfy two criteria derived from the quasi-topological nature of the twisted $(2, 0)$ theory.

First, the compactified theory must be holomorphic along C , and the bulk theory already has this property, so the boundary coupling must also have the same property.

Second, the compactified theory must be independent of the coupling constant e^2 given by the ratio of the radii of the cigars, and the bulk theory is already independent, so the boundary coupling must not introduce a coupling dependence.

The second condition is not as weak as it may sound, because the bulk theory realizes the coupling independence in an interesting manner [26]. Some terms of the bulk action cannot be written in a Q -exact form, apparently leading to a coupling dependence. Still, they fail to be Q -exact by terms quadratic in fermions. So the coupling dependence can actually be absorbed entirely by a rescaling of the fermions. This rescaling makes it hard for the boundary coupling to be supersymmetric (and not Q -exact). Typically, a supersymmetric action contains purely bosonic piece related by supersymmetry to pieces involving fermions, with relative coefficients independent of the coupling. If the bosonic part is coupling independent, then the rescaling introduces a coupling dependence to the fermionic part.

There is a natural candidate satisfying the two criteria: a Chern-Simons term with level independent of the coupling. In fact, the twisted $\mathcal{N} = 4$ super Yang-Mills theory has a Q -invariant complex gauge field

$$\mathcal{A} = (A_x + iY_x)dx + (A_y + iX_y)dy + X_zdz + A_{\bar{z}}d\bar{z}, \quad (4.1)$$

from which one can construct the Chern-Simons action

$$S_{\text{CS}}[\mathcal{A}] = \frac{1}{4\pi i} \int_{L \times C} \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \quad (4.2)$$

for the complexified gauge group $G_{\mathbb{C}}$. Here Tr denotes the Killing form divided by twice the dual Coxeter number h^\vee of G ; for $G = \text{SU}(N)$, it equals the trace in the N -dimensional representation.

However, the Chern-Simons term cannot be all that is present, for it does not lead to the correct boundary conditions. In terms of \mathcal{A} , the Nahm pole boundary conditions (3.9) read

$$\begin{aligned} \mathcal{A}_z &= \frac{t_+}{y} + \dots, \\ \mathcal{A}_{\bar{z}} &= 0, \\ \mathcal{A}_y &= \frac{it_3}{y} + \dots, \end{aligned} \quad (4.3)$$

³Adding boundary couplings modifies the boundary conditions (3.4) and (3.5) which are appropriate for the standard $\mathcal{N} = 4$ super Yang-Mills action. As we will see shortly, however, the added terms are independent of the gauge coupling and so one can take the weak coupling limit where the modified conditions reduce to the original ones.

where $t_+ = t_1 + it_2$. Under variations $\delta\mathcal{A}$, the Chern-Simons action changes by

$$\delta S_{\text{CS}}[\mathcal{A}] = \frac{1}{4\pi i} \int_{L \times C} \text{Tr} \left(2\delta\mathcal{A} \wedge \mathcal{F} - \partial_y (\mathcal{A}_z \delta\mathcal{A}_{\bar{z}} - \mathcal{A}_{\bar{z}} \delta\mathcal{A}_z) dy \wedge dz \wedge d\bar{z} \right). \quad (4.4)$$

The variation is not necessarily zero even if the bulk equations of motion $\mathcal{F} = 0$ is satisfied and the boundary conditions (4.3) are imposed, since $\mathcal{A}_z \delta\mathcal{A}_{\bar{z}}$ may not go to zero as $y \rightarrow 0$ due to the presence of singular terms in \mathcal{A}_z . To fix the problem, we add the term

$$S_{\partial}[\mathcal{A}] = \frac{1}{4\pi i} \int_{L \times C} \text{Tr} \partial_y (\mathcal{A}_z \mathcal{A}_{\bar{z}}) dy \wedge dz \wedge d\bar{z}, \quad (4.5)$$

which would be a boundary term if \mathcal{A}_z were nonsingular. The variation then vanishes up to the bulk equations of motion, provided that $\delta\mathcal{A}$ respects the boundary conditions (4.3) and is regular at $y = 0$.

We therefore propose that the boundary coupling at $y = 0$ is given by a Chern-Simons term (4.2) supplemented with the boundary term (4.5):

$$kS_{\text{CS}+\partial}. \quad (4.6)$$

The level k needs not be an integer. The reason is that Chern-Simons theory on a three-manifold with boundary is required to be gauge invariant only under those gauge transformations that are trivial on the boundary. In our case, these are given by maps $g: L \times C \rightarrow G_{\mathbb{C}}$ such that $g = 1$ at $y = 0$ and ∞ , under which the action changes by the topological invariant

$$\delta S_{\text{CS}+\partial} = -\frac{1}{12\pi i} \int_{L \times C} \text{Tr}(g^{-1} dg)^3. \quad (4.7)$$

If the manifold were closed, δS_{CS} would be a multiple of $2\pi i$ and the level would have to be an integer so that $\exp(-kS_{\text{CS}})$ is gauge invariant. But here, the variation vanishes since L is topologically trivial and so we can continuously deform g to set $g = 1$ everywhere.

At this point one may object that there can be other possibilities for the boundary coupling. In fact, any gauge-invariant boundary terms whose contributions to the energy-momentum tensor are Q -exact (except the component T_{zz}) seem suitable as long as they do not introduce a coupling dependence. However, those terms are not very interesting to us; when placed on a manifold with boundary, they do not induce dynamical degrees of freedom localized on the boundary. The emergence of such boundary degrees of freedom is a consequence of the breaking of gauge invariance by the boundary. We need something that is not gauge invariant, and yet gives a gauge-invariant quantity when integrated over a manifold without boundary.

For Chern-Simons theory, the boundary degrees of freedom are described by a WZW model [22–24]. This can be seen in the path integral formalism as follows [32, 33]. Under gauge transformation

$$\mathcal{A} \rightarrow \mathcal{A}^g = g^{-1} \mathcal{A} g + g^{-1} dg, \quad (4.8)$$

the Chern-Simons action on a three-manifold V with boundary transforms as

$$S_{\text{CS}+\partial}[\mathcal{A}] = S_{\text{CS}+\partial}[\mathcal{A}^g] + S_{\text{gWZW}}[g, \mathcal{A}^g], \quad (4.9)$$

where S_{gWZW} is the following gauged WZW action:

$$S_{\text{gWZW}}[g, A] = \frac{1}{4\pi i} \int_{\partial V} \text{Tr}((g^{-1}\partial g - 2A^{1,0}) \wedge g^{-1}\bar{\partial}g) + \frac{1}{12\pi i} \int_V \text{Tr}(g^{-1}dg)^3. \quad (4.10)$$

Consider the path integral in a neighborhood W of the boundary. The gauge inequivalent configurations of \mathcal{A} in W form a space \mathcal{M}_W , the moduli space of $G_{\mathbb{C}}$ -connections over W . If one makes a gauge choice $\hat{\mathcal{A}}$ at each point of \mathcal{M}_W and defines the Faddeev-Popov determinant Δ by

$$1 = \Delta(\mathcal{A}) \int \mathcal{D}g \delta(\mathcal{A}^g - \hat{\mathcal{A}}), \quad (4.11)$$

then the path integral measure can be written as

$$\int \mathcal{D}\mathcal{A} \exp(-kS_{\text{CS}+\partial}[\mathcal{A}]) = \int_{\mathcal{M}_W} \Delta(\hat{\mathcal{A}}) \int \mathcal{D}g \exp(-k(S_{\text{CS}+\partial}[\hat{\mathcal{A}}] + S_{\text{gWZW}}[g, \hat{\mathcal{A}}])). \quad (4.12)$$

The gauge degrees of freedom are therefore converted on the boundary to dynamical ones, described by the WZW action coupled to the background gauge field $\hat{\mathcal{A}}$.

The action $S_{\text{gWZW}}[g, \hat{\mathcal{A}}]$ coincides with the standard WZW action if one chooses $\hat{\mathcal{A}}_z = 0$ on the boundary, which is always possible as the integrability condition $\partial_A^2 = 0$ is trivially satisfied in complex dimension one. Even so, the emergent boundary theory is generally not the ordinary WZW model due to constraints imposed on gauge transformations by the boundary conditions. A more precise way of saying this is that since the boundary conditions reduce the space of connections and hence the space of boundary gauge transformations required for gauge fixing, the path integral in the definition (4.11) of the Faddeev-Popov determinant should be performed over this reduced space. If the boundary conditions impose first-class constraints, the resulting theory will be a gauged version of the WZW model.

Let us apply the above considerations to our case. In order to gauge away \mathcal{A}_z near $y = 0$, we must allow gauge transformations to be singular at $y = 0$. So we pick some small $\delta > 0$ and integrate over maps $g: [0, \delta) \times C \rightarrow G_{\mathbb{C}}$, including singular ones.⁴ Not all of these maps are equally important, though. The maps that are really relevant are those for which $\mathcal{A}^g = \hat{\mathcal{A}}$. These maps are singular, but we can make a singular gauge transformation so that many of them become regular at $y = 0$ and admit a simple interpretation as boundary degrees of freedom.

Let $\{t_a\}$ be a basis of $\mathfrak{g}_{\mathbb{C}}$ such that $[it_3, t_a] = s_a t_a$, and split $\mathfrak{g}_{\mathbb{C}}$ into the subalgebras \mathfrak{g}_+ , \mathfrak{g}_0 , and \mathfrak{g}_- of s_a positive, zero, and negative. Conjugation by $g_y = \exp(-it_3 \ln y)$ acts on t_a as multiplication by y^{-s_a} . Thus, after the gauge transformation by g_y the boundary

⁴The formula (4.10) still makes sense if we extend g to $L \times C$ and rewrite everything as an integral over $L \times C$ using Stokes' theorem. However, here we must be more careful about what we mean by the path integral over g , since the integrand $\exp(-S_{\text{gWZW}})$ is unbounded due to the noncompactness of $G_{\mathbb{C}}$. (It would be fine to include singular maps otherwise; configurations with diverging kinetic energy simply do not contribute.) The same issue actually arises at the level of Chern-Simons theory, so the method developed in [34] may be adapted to provide a proper path integral definition. Our considerations do not depend on whether or not the path integral formalism is applicable.

condition for \mathcal{A}_z becomes

$$\mathcal{A}_z = t_+ + \sum_{t_a \in \mathfrak{g}_0 \oplus \mathfrak{g}_-} y^{s_a} f_z^a t_a, \quad (4.13)$$

where the coefficient functions f_z^a are less singular than $1/y$. (For \vec{t} giving a principal embedding, the s_a are integers and thus $y^{s_a} f(y) \rightarrow 0$ as $y \rightarrow 0$ if $s_a > 0$ and $f(y)$ is less singular than $1/y$.) Since g_y leaves invariant the action $S_{\text{CS}+\partial}[\mathcal{A}]$, and we want to consider singular gauge transformations anyway, we could as well impose these boundary conditions from the beginning.

The path integral decomposes into different sectors classified by the behavior of the fields at $y = 0$. If we restrict our attention to the sector in which \mathcal{A}_z are regular, then gauge transformations setting \mathcal{A}_z to zero are also regular, as we desired. From now on we will focus on this sector.

The left action $g \mapsto hg$ by maps $h(z)$, holomorphic along C and constant along L , would be a symmetry of the boundary theory if the theory were the ordinary WZW model. The usual story is that this symmetry implies the existence of an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ of level k in the chiral algebra. (The antiholomorphic counterpart of this algebra is trivial in Q -cohomology.) Here we get a smaller algebra because of the boundary conditions. Indeed, the gauge-fixing condition $\mathcal{A}_z^g = 0$ relates the affine currents $J = J_z dz$ to the boundary value of \mathcal{A}_z :

$$J_z = -k\partial_z gg^{-1}|_{y=0} = k\mathcal{A}_z|_{y=0}. \quad (4.14)$$

Comparing this formula with the boundary condition (4.13), we see that the affine currents obey

$$\sum_{t_a \in \mathfrak{g}_+} J^a t_a - kt_+ = 0. \quad (4.15)$$

This equation encodes first-class constraints; the left-hand side generates the gauge transformations by the subgroup G_- of $G_{\mathbb{C}}$ with Lie algebra \mathfrak{g}_- . Therefore, the boundary degrees of freedom are described by a G_- -gauged WZW model.

To summarize, we have argued that the boundary coupling on $\{0\} \times L_2 \times C$ is given by a Chern-Simons term with the Nahm pole boundary conditions (4.3), which induce dynamical degrees of freedom on $\{0\} \times \{0\} \times C$ described by the above gauged WZW model. On the other boundary $L_1 \times \{0\} \times C$, we have dual boundary coupling with dual boundary condition. In particular, it gives a dual description of the two-dimensional degrees of freedom.

5 The AGT correspondence

The connection to the AGT correspondence is now clear. The BRST cohomology represented by the constrained affine currents is the W-algebra $\mathcal{W}_k(\mathfrak{g})$, obtained from $\widehat{\mathfrak{g}}$ of level k by quantum Drinfeld-Sokolov reduction [35] with respect to the principal embedding specified by the Nahm pole. This is exactly the symmetry that one finds on the two-dimensional side of the AGT correspondence.

The level k can be fixed by comparing the two effective descriptions of the partition function of the $(2, 0)$ theory, namely the Nekrasov partition function of the $\mathcal{N} = 2$ theory

and the relevant conformal block of the gauged WZW model. This was the original idea of [1], and leads to the identification

$$k = -h^\vee - \frac{\epsilon_2}{\epsilon_1}. \quad (5.1)$$

Notice that the six-dimensional description is symmetric under the exchange of ϵ_1 and ϵ_2 , which amounts to S -duality in $\mathcal{N} = 4$ super Yang-Mills theory. So the same must be true for the resulting W-algebra. There is indeed an isomorphism [36]

$$\mathcal{W}_k(\mathfrak{g}) \cong \mathcal{W}_{k'}({}^L\mathfrak{g}) \quad (5.2)$$

for any simple Lie algebra \mathfrak{g} , with the levels related by $k + h^\vee = (k' + h^\vee)^{-1}$. Here ${}^L\mathfrak{g}$ is the Langlands dual of \mathfrak{g} , which in the simply-laced case is the same as \mathfrak{g} .

As an example, take $\mathfrak{g} = A_1$. In this case the gauged WZW model is Liouville theory and the W-algebra in question is the Virasoro algebra. More generally, for $\mathfrak{g} = A_{N-1}$, one gets a Toda theory with symmetry algebra \mathcal{W}_N , which contains the Virasoro algebra with central charge

$$c = N - 1 + (N^3 - N) \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}. \quad (5.3)$$

The central charge exhibits the N^3 scaling behavior of the entropy of the $(2, 0)$ theory [37], reflecting the six-dimensional origin of the two-dimensional degrees of freedom.

Let us see how the W-algebra structure is encoded in the physics of the $\mathcal{N} = 2$ theory. To simplify the analysis, we content ourselves with the semiclassical approximation which is good when k is large. Using the G_- gauge symmetry one can put the affine currents into the form

$$J_z/k = t_+ + \sum_{t_i \in \ker(t_-)} W_i t_i, \quad (5.4)$$

where the sum is over the t_a in the kernel of the adjoint action by $t_- = t_1 - it_2$. For $\mathfrak{g} = A_{N-1}$, the gauged-fixed form of J is

$$J_z/k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ W_N & W_{N-1} & W_{N-2} & \dots & W_2 & 0 \end{pmatrix}, \quad (5.5)$$

The classical W-algebra is generated by the currents W_i .

We consider the following polynomial in x :

$$\det(x - J_z/k). \quad (5.6)$$

Its coefficients are elementary symmetric polynomials in the eigenvalues of J_z , and as such can also be written as polynomials in the Casimir operators $\text{Tr } J_z^i$, $i = 2, \dots, \text{rank } \mathfrak{g}$. The $\mathcal{N} = 2$ theory compactified down to two dimensions using cigars is equivalent to the $\mathcal{N} = 4$

super Yang-Mills theory we have been studying, compactified further on C . The adjoint scalar ϕ from the vector multiplet of the $\mathcal{N} = 2$ theory is identified with X_z . Noting that $J_z/k = X_z$ in the Q -cohomology since this holds at $x = y = 0$, we see that the Casimirs are identified with the generators $\text{Tr } \phi^i$ of the chiral ring of the $\mathcal{N} = 2$ theory. The coefficients of the polynomial (5.6) can of course be expressed in terms of the W_i , so we obtain a relation between the W-currents and these generators. For example, for $\mathfrak{g} = A_{N-1}$ we have

$$\det(x - J_z/k) = x^N - \sum_{i=2}^N W_i x^{N-i}, \quad (5.7)$$

and we find the correspondence

$$W_i \sim \text{Tr } \phi^i + \dots. \quad (5.8)$$

Essentially the same relation was proposed in [9].

There is a more refined correspondence, relating the W-currents to the Seiberg-Witten curve Σ of the $\mathcal{N} = 2$ theory which is a branched cover of C . If we choose coordinates (x, z) for the holomorphic cotangent bundle of C , then Σ is given [14] in the absence of the Ω -deformation by

$$\langle \det(x - X_z) \dots \rangle = \left(x^N + \sum_{i=2}^N u_i(z) x^{N-i} \right) \langle \dots \rangle = 0, \quad (5.9)$$

where the ellipses denote the relevant defect operators inserted at the punctures of C . (The presence of the boundaries can be neglected if X_z is placed far away from them.) The restriction of the one-form xdz to Σ is the Seiberg-Witten differential. Evaluating the correlation function at $x = y = 0$, we find

$$\langle W_i \dots \rangle \sim u_i \langle \dots \rangle \quad (5.10)$$

in the undeformed limit. Hence, the expectation values of the W-currents determine the Seiberg-Witten curve, from which the Coulomb branch parameters can be read off. This relation was conjectured in [1], and has been checked in [38, 39].

We have seen that our approach explains important aspects of the AGT correspondence, that is, how W-algebras arise from the $(2, 0)$ theory in the Ω -background and how they are related to the physics of $\mathcal{N} = 2$ gauge theories obtained by compactification on Riemann surfaces. Our construction can be generalized in a number of ways.

Clearly, it is desirable to treat all gauge groups in a uniform fashion, not just simply-laced ones. $\mathcal{N} = 2$ theories with non-simply-laced gauge groups can be constructed from the $(2, 0)$ theory by compactification on Riemann surfaces with outer automorphism twists [40], and there seems to be no obstacle to adapting our construction to this situation. The symmetry algebras of the conformal field theories will then be not ordinary W-algebras but twisted ones, and associated to the Langlands dual of the affine Kac-Moody algebra of the gauge group [7, 39].

Another possibility is to replace \mathbb{R}^4 by an ALE orbifold $\mathbb{C}^2/\mathbb{Z}_k$. In this case the symmetry algebras are parafermionic W-algebras for $k > 1$ [19, 41]. The main problem here will

be to identify the boundary conditions. Once that is done, the same argument should lead to a BRST construction of these algebras, generalizing quantum Drinfeld-Sokolov reduction in the $k = 1$ case.

Finally, one may include a half-BPS surface operator in the $\mathcal{N} = 2$ theory side [5–8]. This situation can be realized in our construction by placing a codimension-two defect at the origin of D_2 . The presence of the defect changes the residue of the Nahm pole to another one corresponding to a different $\mathfrak{su}(2)$ embedding [42, 43], hence the resulting W-algebra to the one associated to this new embedding. So everything we have said about the W-algebra simply carries over to this case, except one important point: the duality of W-algebras is lost, since the setup is no longer symmetric between D_1 and D_2 . To remedy the asymmetry, we can place another defect at the origin of D_2 . Then the setup is symmetric again, under the exchange of ϵ_1 and ϵ_2 together with the exchange of the defects. This consideration suggests that there is a generalization of quantum Drinfeld-Sokolov reduction whose data are a Lie algebra and two $\mathfrak{sl}(2)$ embeddings, such that it enjoys an analogous duality and reduces to the ordinary quantum Drinfeld-Sokolov reduction when one of the embeddings is principal.

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